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# Almost Sure Quasilocality Fails for the Random-Cluster Model on a Tree

**Olle Häggström**[1]

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We study the random-cluster model on a homogeneous tree, and show that the following three conditions are equivalent for a random-cluster measure: quasilocality, almost sure quasilocality, and the almost sure nonexistence of infinite clusters. As a consequence of this, we find that the plus measure for the Ising model on a tree at sufficiently low temperatures can be mapped, via a local stochastic transformation, into a measure which fails to be almost surely quasilocal.

**KEY WORDS:** Random-cluster model; quasilocality; almost sure quasilocality; tree; Gibbs measure; Ising model.

## 1. INTRODUCTION

The concept of almost sure quasilocality has recently received a considerable amount of attention in statistical mechanics. The reason for this is that it has become increasingly apparent that many systems of physical interest fail to be quasilocal; see, e.g., refs. 12, 20, 6, 7, 5, and 4. In particular, the class of Gibbs measures is not closed under various renormalization transformations, and many examples where quasilocality fails for the renormalized systems can be found in the above references. This behavior is in general undesirable, and therefore such renormalizations are often described as "pathological." This state of affairs makes it natural to try to find some useful weakening of the concept of a Gibbs measure, and one such approach, suggested by Fernández and Pfister,[9] is to replace the condition of quasilocality by the weaker notion of almost sure quasilocality (see also refs. 19, 17, and 8).

The purpose of this note is to show that, in general, even this weaker condition fails for the random-cluster model on an infinite homogeneous

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[1] Department of Mathematics, Chalmers University of Technology, 412 96 Göteborg, Sweden.

tree of order $n \geqslant 2$, studied in ref. 15. In the case of the random-cluster model on $\mathbf{Z}^d$, almost sure quasilocality was shown (at least for translation-invariant random-cluster measures) by Grimmett[14] and Pfister and Vande Velde.[19] Hence, this note may be viewed as a contribution to the tradition of showing that a random process on $\mathbf{Z}^d$ which behaves well, in some specified sense, fails to do so when $\mathbf{Z}^d$ is replaced by a tree; see, e.g., refs. 21, 18, 2, and 15.

Our results imply the existence of a local transformation of the Ising model on a tree at low temperature which is pathological in a rather strong sense, in that it brings the system not only outside of the class of quasilocal systems, but also outside of the class of almost surely quasilocal systems. Moreover, we find that this transformation applied to two different Gibbs measures $\nu_f$ and $\nu_+$ for the same potential can lead to a quasilocal measure for $\nu_f$ and a nonquasilocal (not even almost surely quasilocal) measure for $\nu_+$.

Another example of a transformation which brings a Gibbsian system outside of the class of almost surely quasilocal systems was recently given by van Enter and Lőrinczi[8] (building on work by Lőrinczi and Vande Velde).[16] Their example is a variant of Schonmann's[20] projected Ising model.

The random-cluster model is defined in Section 2. In Section 3, we give the definitions of quasilocality, resp. almost sure quasilocality, and state and prove the main result, which characterizes which random-cluster measures have the almost sure quasilocality property. In Section 4, we discuss the relation with the Ising model and, in particular, describe the transformation which takes the Ising model to the random-cluster model and which has the pathological behavior described above.

## 2. THE RANDOM-CLUSTER MODEL

The random-cluster model (see refs. 10, 1, 3, and 13 for further background) with parameters is $p \in [0, 1]$ and $q > 0$ is easiest to define on a finite graph $G$ with vertex set $V$ and edge set $E$. By a subgraph of $G$ we mean a graph with the same vertex set $V$ as $G$ and an edge set which is a subset of $E$. Such a subgraph is identified with an element of $\{0, 1\}^E$, where a 1 indicates that an edge is present and a 0 indicates that it is absent.

**Definition 2.1.** The *random-cluster measure* $\mu_G^{p,q}$ for $G$ with parameters $p$ and $q$ is the probability measure on the set of subgraphs of $G$ given by

$$\mu_G^{p,q}(\eta) = \frac{1}{Z_G^{p,q}} \left\{ \prod_{e \in E} p^{\eta(e)} (1-p)^{1-\eta(e)} \right\} q^{k(\eta)}$$

for all $\eta \in \{0, 1\}^E$. Here $k(\eta)$ is the number of connected components of $\eta$ and

$$Z_G^{p,q} = \sum_{\eta \in \{0,1\}^E} \left\{ \prod_{e \in E} p^{\eta(e)} (1-p)^{1-\eta(e)} \right\} q^{k(\eta)}$$

is a normalizing constant.

Let $\mathbf{T}_n$ be the homogeneous tree of order $n$, i.e., $\mathbf{T}_n$ is the (unique) infinite graph which is connected, has no circuits, and has $n+1$ branches emanating from every vertex. We will always assume that $n \geqslant 2$; the tree obtained with $n=1$ is simply the nearest neighbor graph on $\mathbf{Z}$, and our results do not apply to this case. Write $\mathbf{V}_n$ and $\mathbf{E}_n$ for the vertex set and the edge set, respectively, of $\mathbf{T}_n$.

Note that Definition 2.1 is not applicable to infinite graphs like $\mathbf{T}_n$, because the number of connected components (clusters) will in general be infinite. Instead, the following definition, analogous to the Dobrushin–Lanford–Ruelle definition of a Gibbs measure, was given in ref. 15. Given $\Lambda \subset \mathbf{E}_n$, define $\Lambda' \subset \mathbf{V}_n$ to be the set $\{v \in \mathbf{V}_n \colon \exists e \in \Lambda$ such that $e$ is incident to $v\}$. For a configuration $\xi \in \{0, 1\}^{\mathbf{E}_n \setminus \Lambda}$, let

$$\mu_{\Lambda,\xi}^{p,q}(\eta) = \frac{1}{Z_{\Lambda,\xi}^{p,q}} \left\{ \prod_{e \in \Lambda} p^{\eta(e)} (1-p)^{1-\eta(e)} \right\} q^{k(\eta,\xi)} \tag{1}$$

for all $\eta \in \{0, 1\}^\Lambda$, where $k(\eta, \xi)$ is the number of *finite* connected components which intersect $\Lambda'$ in the configuration which agrees with $\eta$ on $\Lambda$ and with $\xi$ on $\mathbf{E}_n \setminus \Lambda$, and $Z_{\lambda,\xi}^{p,q}$ of course again is the right normalizing constant.

**Definition 2.2.** A probality measure $\mu$ on $\{0, 1\}^{\mathbf{E}_n}$ is called a random-cluster measure with parameters $p$ and $q$ if its conditional probabilities satisfy

$$\mu(\eta \mid \xi) = \mu_{\Lambda,\xi}^{p,q}(\eta)$$

for all finite $\Lambda \subset \mathbf{E}_n$, all $\eta \in \{0, 1\}^\Lambda$, and $\mu$-a.e. $\xi \in \{0, 1\}^{\mathbf{E}_n \setminus \Lambda}$. Here $\mu_{\Lambda,\xi}^{p,q}(\eta)$ is given by (1).

This is slightly different from the definition in ref. 14 of a random-cluster measure for $\mathbf{Z}^d$, the difference being that $k(\eta, \xi)$ counts only the finite connected components intersecting $\Lambda'$, rather than all connected components intersecting $\Lambda'$. One may ask why we use this modified way of counting connected components on a tree, and there are three answers to this question:

1. It is believed that the number of infinite clusters a.s. is either 0 or 1 for the random-cluster model on $\mathbf{Z}^d$ (this is certainly the case if we restrict to translation invariant measures; see ref. 14). If this is true, then it is easy to see that it would lead to an equivalent definition on $\mathbf{Z}^d$ to count finite clusters rather than all clusters. Also, the two ways of counting clusters are (trivially) the same on a finite graph.

2. On a tree, the absence of circuits implies that adding an edge always reduces the number of clusters by exactly 1. Therefore an edge $e \in \mathbf{E}_n$ would be present with probality $p(p+(1-p)q)^{-1}$ independently of all other edges if $k(\eta, \xi)$ were to count all clusters, so that the random-cluster model would be nothing more than a complicated way of defining i.i.d. measure on $\{0, 1\}^{\mathbf{E}_n}$.

3. The random-cluster model on $\mathbf{T}_n$ as we define it, with $q=2$, is intimately related to the Ising model on $\mathbf{T}_n$ with the 'plus' boundary condition. A similar statement holds for the Potts model with $q \in \{3, 4,...\}$. This relation was not discussed explicitly in ref. 15, so we will do so in Section 4. (To be fair, we should also mention that the alternative definition of the random cluster on $\mathbf{T}_n$ bears the same relation to the Ising/Potts models with the 'free' boundary condition.)

We refer to ref. 15 for further discussion and results on this model.

## 3. MAIN RESULT

Let $A$ be a finite set, $S$ a countable set, and let $\Omega = A^S$ (below, we will take $A = \{0, 1\}$ and $S = \mathbf{E}_n$). For $\Lambda \subseteq S$ and $\omega \in \Omega$, write $\omega_\Lambda \in A^\Lambda$ for $\omega$ restricted to $\Lambda$. We say that a function $g$ on $\Omega$ is *quasilocal* at the point $\xi \in \Omega$ if for any $\varepsilon > 0$ there exists a finite set $\Lambda_\varepsilon \subset S$ such that

$$\sup_{\xi' :\, \xi'_{\Lambda_\varepsilon} = \xi_{\Lambda_\varepsilon}} |g(\xi) - g(\xi')| < \varepsilon$$

**Definition 3.1.** A probability measure $P$ is said to be *quasilocal* if for any finite $\Lambda \subset S$ it admits a conditional distribution $P(\cdot \mid \cdot)$ of the configuration on $\Lambda$ given the configuration on $S \backslash \Lambda$ such that for any $\eta \in A^\Lambda$ and any $\xi \in \Omega$ the function

$$P(\eta \mid \xi_{S \backslash \Lambda})$$

is quasilocal at $\xi$. We say that $P$ is *almost surely quasilocal* if it admits a conditional distribution such that this holds for all $\eta \in A^\Lambda$ and $P$-a.e. $\xi \in \Omega$.

Note that quasilocality implies almost sure quasilocality. An i.i.d. measure on $\Omega$ is obviously quasilocal, whence a random-cluster measure for $\mathbf{T}_n$ whose parameters satisfy either $p \in \{0, 1\}$ or $q = 1$ is quasilocal. For other parameter values, the situation is described in the following theorem, which is our main result.

**Theorem 3.2.** Let $\mu$ be a probability measure on $\{0, 1\}^{\mathbf{E}_n}$, and suppose that $\mu$ is a random-cluster measure with parameters $p \in (0, 1)$ and $q \neq 1$. Let $C$ be the event that these exists at least one infinite cluster. Then the following three conditions are equivalent.

(i) $\mu(C) = 0$

(ii) $\mu$ is quasilocal.

(iii) $\mu$ is almost surely quasilocal.

Some results on whether a given random-cluster measure assigns positive probability to the existence of infinite clusters are given in ref. 15 (see also Section 4). In fact, for certain values of the parameters $p$ and $q$, these exists both a random-cluster measure which assigns probability 0 to the existence of infinite clusters and one for which this event has probability 1.

*Proof of Theorem 3.2.* Since quasilocality implies almost sure quasilocality, it suffices to show (i) ⇒ (ii) and (iii) ⇒ (i). We pick an edge $e \in \mathbf{E}_n$ and consider the conditional probability $\mu(e \text{ is present } | \xi_{\mathbf{E}_n \setminus \{e\}})$. Let $C_e^*$ be the event that both endvertices of $e$ have paths to infinity that do not use $e$. It is immediate from Definition 2.2 that $\mu$-a.s.

$$\mu(e \text{ is present } | \xi_{\mathbf{E}_n \setminus \{e\}}) = \begin{cases} p & \text{on } \quad C_e^* \\ \dfrac{p}{p + (1-p)\, q} & \text{on } \quad \neg C_e^* \end{cases} \tag{2}$$

(the complement of an event $A$ is denoted $\neg A$). Note that $p \neq p(p + (1-p)\, q)^{-1}$ by the assumptions of the theorem.

In order to prove (i) ⇒ (ii), suppose that $\mu(C) = 0$. Clearly $\neg C$ implies $\neg C_e^*$ for any $e$. Hence, by (2), each edge $e$ is present with probability $p(p + (1-p)\, q)^{-1}$ independently of everything else, so $\mu$ is simply an i.i.d. measure on $\{0, 1\}^{\mathbf{E}_n}$, whence it is quasilocal.

To prove (iii) ⇒ (i), we will show that $\mu(C) > 0$ implies that $\mu$ cannot be almost surely quasilocal. Suppose first that $\mu(C_e^*)$ has positive probability for some $e$. For any finite $\Lambda$ containing $e$ and any $\xi \in C_e^*$, it is possible to find a larger, but finite, edge set $\Lambda' \supset \Lambda$ such that by erasing all edges of $\xi$ in $\Lambda' \setminus \Lambda$ we obtain a configuration which is not in $C_e^*$ (just take $\Lambda'$ to consist of $\Lambda$ plus the edges of $\mathbf{E}_n \setminus \Lambda$ that are adjacent to $\Lambda$). It follows from Definition 2.2 that every configuration on $\Lambda'$ has positive $\mu$-probability; in particular, this holds for the configuration which agrees with $\xi$ on $\Lambda$ and which is identically zero on $\Lambda'\backslash\Lambda$. Hence $\mu(e$ is present $|\xi_{\mathbf{E}_n}\backslash\{e\})$, viewed as a function of $\xi$, cannot be quasilocal for any version of the conditional probabilities $\mu(\cdot|\cdot)$, and moreover $\mu$ cannot be almost surely quasilocal.

It remains to show that $\mu(C)>0$ implies that $\mu(C_e^*)>0$ for some $e\in\mathbf{E}_n$. Designate some vertex $v\in\mathbf{V}_n$ to be the root of $\mathbf{T}_n$, and write $\Lambda_k$ for the set of edges having both endvertices within distance $k$ from the root. Also write $\pi$ for and *end* of $\mathbf{T}_n$; by an end we mean an infinite self-avoiding path starting at the root. For a given $\xi\in\{0,1\}^{\mathbf{E}_n}$, call an end $\pi$ *open* if all but at most finitely many edges in $\pi$ are present in $\xi$. Let $C^*$ denote the event that there exist at least two different open ends. We now claim that

$$\mu(C)>0\Rightarrow\mu(C^*)>0 \tag{3}$$

or equivalently that $\mu(C^*)=0$ implies $\mu(C)=0$. To show this, suppose that $\mu(C^*)=0$. Then (2) implies that $\mu$ is i.i.d. measure with edge probability $p(p+(1-p)\,q)^{-1}$. By Kolmogorov's 0–1 law, the number of open ends is then an almost sure constant $c$. A simple branching process comparison shows that

$$c=\begin{cases}0 & \text{if}\quad np/[p+(1-p)\,q]\leqslant 1\\ \infty & \text{otherwise}\end{cases}$$

so that in the former case we have that $\mu(C)=0$ and in the latter case we have a contradiction to the assumption that $\mu(C^*)=0$. Hence (3) is established. [With slightly more work, one can show the stronger fact that $\mu(C^*)=\mu(C)$, but we do not need this.]

Next, we observe that $\mu(C^*)>0$ implies that the existence of a doubly infinite path in $\xi$ has positive probability (to see this, pick $k$ so large that with positive probability there are two open ends which are disjoint outside $\Lambda_k$ and whose edges in $\mathbf{E}_n\backslash\Lambda_k$ are all present, and note that they connect via $\Lambda_k$ with positive probability). The existence of a doubly infinite path in $\xi$ in turn implies that $C_e^*$ occurs for some $e$. Putting things together, we have

$$\mu(C)>0\Rightarrow\mu(C^*)>0\Rightarrow\exists e \text{ such that } \mu(C_e^*)>0$$

so the proof is complete. ∎

**Remark.** In the proof of (i) ⇒ (ii), the specification of $\mu(\cdot|\cdot)$ which we implicitly use to show quasilocality is that the edges on a finite set $\Lambda$ are i.i.d. with edge probability $p(p+(1-p)\,q)^{-1}$, regardless of the configuration on $\mathbf{E}_n\backslash\Lambda$. This is different from the specification in (1), which defines a random-cluster measure (of course, these specifications differ only on a set of $\mu$-measure 0). The specification in (1) is not quasilocal, but we could have used it if we were content with showing (i) $\Rightarrow$ (iii).

## 4. RELATION WITH THE ISING MODEL

In this section, we show how a certain random-cluster measure for $\mathbf{T}_n$ can be obtained by a simple stochastic transformation of the "plus" measure for the Ising model on $\mathbf{T}_n$. The same thing can of course also be done for the "minus" measure, and also for the $q$ ordered states of the Potts model with $q \geqslant 3$. We restrict consideration to the Ising model for simplicity and because it suffices for our purpose, which is to give an example of a local transformation of a Gibbs measure which results in a measure which fails to be almost surely quasilocal.

The Ising model (see ref. 11 for a general treatment) with reciprocal temperature $\beta > 0$ on a finite graph $G$ with vertex set $V$ and edge set $E$ is defined as follows. A configuration $\omega \in \{-1, 1\}^G$ has energy

$$H(\omega) = -\beta \sum_{\langle x, y \rangle} \omega(x)\, \omega(y)$$

where the sum is taken over all pairs of vertices $x$ and $y$ that have an edge connecting them. The Gibbs measure $\nu_G^\beta$ on $\{-1, 1\}^G$ for the Ising model at temperature $\beta^{-1}$ is the probability measure for which

$$\nu_G^\beta(\omega) = \frac{1}{Z_G^\beta} \exp\{-H(\omega)\}$$

for all $\omega \in \{-1, 1\}^G$. Here $Z_G^\beta$ is a normalizing constant. Replacing $G$ by the infinite graph $\mathbf{T}_n$, we need to define Gibbs measures in terms of conditional probabilities. We call a measure $\nu$ on $\{-1, 1\}^{\mathbf{V}_n}$ a Gibbs measure for the Ising model at temperature $\beta^{-1}$ if for any finite set $L \subset \mathbf{V}_n$, any $\omega \in \{-1, 1\}^L$, and $\nu$-a.e. $\omega' \in \{-1, 1\}^{\mathbf{V}_n \setminus L}$ we have

$$\nu(\omega \mid \omega') = \frac{1}{Z_{L, \omega'}^\beta} \exp\{-H(\omega, \omega')\} \tag{4}$$

where $Z_{L, \omega'}^\beta$ is again the appropriate normalizing constant, and

$$H(\omega, \omega') = -\sum_{\langle x, y \rangle} \omega(x)\, \omega'(y)$$

where this time the sum is taken over all pairs of vertices $(x \in L, y \in \mathbf{V}_n)$ that have an edge connecting them. Let $L_k \subset \mathbf{V}_n$ be the set of vertices within distance $k-1$ from the root, and define $\nu^{\beta}_{+,k}$ to be the probability measure on $\{-1, 1\}^{\mathbf{V}_n}$ which assigns probability 1 to the configuration $\omega_+ \equiv 1$ on $\mathbf{V}_n \backslash L_k$, and whose projection on $L_k$ is given by (4) with $\omega' = \omega_+$. It is well known that the measure

$$\nu^{\beta}_{+} = \lim_{k \to \infty} \nu^{\beta}_{+,k} \tag{5}$$

exists and is a Gibbs measure.

There is an analogous measure for the random-cluster model. As before, let $\Lambda_k \subset \mathbf{E}_n$ be the set of edges that have both endvertices within distance $k$ from the root. Given the parameters $p \in (0, 1)$ and $q \geqslant 1$, let $\mu^{p,q}_{1,k}$ be the measure on $\{0, 1\}^{\mathbf{E}_n}$ which assigns probability 1 to the configuration $\xi_1 \equiv 1$ on $\mathbf{E}_n \backslash \Lambda_k$ and whose projection on $\Lambda_k$ is $\mu^{p,q}_{\Lambda_k, \xi_1}$. The limiting measure

$$\mu^{p,q}_{1} = \lim_{k \to \infty} \mu^{p,q}_{1,k} \tag{6}$$

exists and is a random-cluster measure.[15] We may think of $\mu^{p,q}_1$ as obtained with "wired" boundary conditions, because the way we count connected components for $\mu^{p,q}_{1,k}$ is equivalent to viewing everything outside $\Lambda_k$ as a single connected component.

Proposition 4.1 below gives a relation between $\nu^{\beta}_{+}$ and $\mu^{p,q}_1$ analogous to the Edwards–Sokal coupling of the corresponding measures $\nu^{\beta}_{G}$ and $\mu^{p,q}_{G}$ for finite graphs.[3,13] Let $\tilde{\mu}^{\beta}_{+}$ be the probability measure on $\{0, 1\}^{\mathbf{E}_n}$ obtained as follows. First, pick a configuration $\omega \in \{-1, 1\}^{\mathbf{V}_n}$ according to $\nu^{\beta}_{+}$. Conditional on $\omega$, assign each edge $e \in \mathbf{E}_n$ with endvertices $x, y \in \mathbf{V}_n$ value 1 with probability

$$\theta(e, \omega) = \begin{cases} p & \text{if} \quad \omega(x) = \omega(y) \\ 0 & \text{otherwise} \end{cases}$$

and value 0 with probability $1 - \theta(e, \omega)$, and do this independently for each $e$.

**Proposition 4.1.** Tacking $q = 2$ and $p = 1 - e^{-2\beta}$, we have

$$\tilde{\mu}^{\beta}_{+} = \mu^{p,q}_{1}$$

For $q = 2, \mu^{p,q}_1$ assigns positive probability to the existence of infinite clusters iff $p > 2(n+1)^{-1}$ (see ref. 15; in fact, it then assigns probability one to the existence of infinitely many infinite clusters). By Theorem 3.2, we thus have a local stochastic transformation for $\nu_+^\beta$ with the pathological behavior described above, whenever

$$\beta > \frac{1}{2} \log \left( \frac{n+1}{n-1} \right)$$

*Proof.* Let $P'_k$ be the probability measure on $\{-1, 1\}^{\mathbf{V}_n} \times \{0, 1\}^{\mathbf{E}_n}$ obtained as follows. Each vertex $v \in \mathbf{V}_n$ takes value 1 with probability

$$\begin{cases} \frac{1}{2} & \text{if } \; v \in L_k \\ 1 & \text{otherwise} \end{cases}$$

each edge $e \in \mathbf{E}_n$ takes value 1 with probability

$$\begin{cases} p & \text{if } \; e \in \Lambda_k \\ 1 & \text{otherwise} \end{cases}$$

and this is done independently for all vertices and edges. Next, let the probability measure $P_k$ be equal to $P'_k$ conditioned on the event that no two vertices with different values have an edge connecting them. It is now easy to check that the marginals of $P_k$ on $\{-1, 1\}^{\mathbf{V}_n}$ and $\{0, 1\}^{\mathbf{E}_n}$ are $\nu_{+,k}^\beta$ and $\mu_{1,k}^{p,q}$, respectively. [This is in fact exactly the Edwards–Sokal coupling of $\nu_G^\beta$ and $\mu_G^{p,q}$, where $G$ is the finite graph with vertex set $L_k \cup \{v^*\}$ and edge set $\Lambda_k$, conditioned on the event that $v^*$ takes value 1. Here $v^*$ is the vertex obtained by collapsing all vertices of $\mathbf{V}_n \backslash L_k$ into a single vertex.] It is immediate from the construction of $P_k$ that, conditional on $\omega \in \{0, 1\}^{\mathbf{V}_n}$, each edge $e \in \Lambda_k$ is present with probability $\theta(e, \omega)$ independently of all other edges. This observation, together with (5) and (6), implies that

$$P = \lim_{k \to \infty} P_k$$

exists and has marginals $\nu_+^\beta$ and $\mu_1^{p,q} = \tilde{\mu}_+^\beta$. ▌

We now consider another Gibbs measure $\nu_f^\beta$ for the Ising model at the same temperature $\beta^{-1}$. We obtain it with "free" boundary conditions, in the following way. Define the probability measure $\nu_{f,k}^\beta$ on $\{-1, 1\}^{\mathbf{V}_n}$ arbitrarily on $\mathbf{V}_n \backslash L_k$, and let it be given by $\nu_G^\beta$ on $L_k$, where $G$ here is the graph with vertex set $L_k$ and edge set $\Lambda_{k-1}$. As in (5), the limiting measure

$$\nu_f^\beta = \lim_{k \to \infty} \nu_{f,k}^\beta$$

exists and is a Gibbs measure. It is well known that both $\nu_f^\beta$ and $\nu_+^\beta$ are invariant under graph automorphisms of $\mathbf{T}_n$.

Now let $\tilde{\mu}_f^\beta$ be the measure on $\{0, 1\}^{\mathbf{E}_n}$ obtained from $\nu_f^\beta$ in the same way as $\tilde{\mu}_+^\beta$ was obtained from $\nu_+^\beta$. It turns out that $\tilde{\mu}_f^\beta$ is simply i.i.d. measure with edge probability $p(p+(1-p)\,q)^{-1}$; the proof of this goes exactly as the proof of Proposition 4.1. For

$$\beta \leqslant \frac{1}{2}\log\left(\frac{n+1}{n-1}\right)$$

the measures $\nu_f^\beta$ and $\nu_+^\beta$ coincide, but for

$$\beta > \frac{1}{2}\log\left(\frac{n+1}{n-1}\right)$$

what we have is a local stochastic transformation which, applied to the two automorphism-invariant Gibbs measures $\nu_f^\beta$ and $\nu_+^\beta$ defined for the same potential, gives a quasilocal measure ($\tilde{\mu}_f^\beta$) in one case and a nonquasilocal measure ($\tilde{\mu}_+^\beta$) in the other. It follows from Theorem 3.4 in ref. 7 that no such monkey business can happen on $\mathbf{Z}^d$.

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